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An Application of Fractional Calculus in Solving Generalized New Fractional Kinetic Equation

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Abstract:
In this paper, we derive the solution of New generalized fractional kinetic equation involving the New generalized Miller-Ross function. The result obtained here is quite general in nature and capable of yielding a very large number of results. Special cases, involving the M-Series, Mittag-Leffler function and Miller-Ross function etc. are also considered.

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1. Introduction:
The great importance of mathematical physics in distinguished astrophysical problems has attracted astronomers, mathematician and physicists to pay more attention to available mathematical tools that can be widely used in solving several problems of physics, applied physics and astrophysics. A spherically symmetric non-rotating, self-gravitating model of star like the Sun is assumed to be in thermal equilibrium and hydrostatic equilibrium. The star is characterized by its mass, luminosity, effective surface temperature, radius, central density and central temperature. The stellar structures and their mathematical models are investigated on the basis of above characters and some additional information related to the equation of state, nuclear energy generation rate and the opacity. The assumptions of thermal equilibrium and hydrostatic equilibrium imply that there is no time dependence in the equations describing the internal structure of the star (Kourganoff [11], Perdang [17] and Clayton [2]). Energy in such stellar structures is being produced by the process of chemical reactions (thermonuclear reactions).

Consider an arbitrary reaction characterized by a time dependent quantity $N = N(t)$. It is possible to calculate rate of change dN/dt to a balance between the destruction rate d and the production rate p of N, that is $\frac{dN}{dt} = -d + p$.In general, through feedback or other interaction mechanism, destruction and production depend on the quantity N itself: $d = d(N)\ or\ p = p(N)$. This dependence is complicated since the destruction or production at time t depends not only on N(t) but also on the past history N(τ), τ < t, of the variable N.

This may be formally represented by (Haubold and Mathai [7])

$$dN/dt = -d\,(Nt) + p(Nt), \tag{1}$$
where Nt denotes the function defined by $Nt(t*) = N(t - t*), t* > 0$.

Haubold and Mathai [7] studied a special case of this equation, when spatial fluctuations or inhomogenities in quantity N(t) are neglected, is given by the equation

$$\frac{dN_i}{dt} = -c_i N_i(t) \tag{2}$$

with the initial condition that $Ni\ (t = 0) = N_0$ is the number density of species i at time $t = 0$; constant $c_i > 0$, known as standard kinetic equation.
The solution of the equation (2) is given by

$$Ni(t) = N_0\, e^{-c_i t} \tag{3}$$

An alternative form of the same equation can be obtained on integration:

$$N(t) - N_0 = c\, {}_0D_t^{-1}\, N(t) \tag{4}$$

where ${}_0D_t^{-1}$ is the standard integral operator. Haubold and Mathai [7] have given the fractional generalization of the standard kinetic equation (2) as

$$N(t) - N_0 = c^v\, {}_0D_t^{-v}\, N(t) \tag{5}$$

where ${}_0D_t^{-v}$ is the well known Riemann-Liouville fractional integral operator (Oldham and Spanier 16]; Samko et al. [19]; Miller and Ross [13]) defined by

$${}_0D_t^{-v}N(t) = \frac{1}{\Gamma(v)}\int_0^t (t - u)^{v-1} f(u), \quad R(v) > 0, \tag{6}$$

The solution of the fractional kinetic equation (6) is given by (see Haubold and Mathai [7])

$$N(t) = N_0 \sum_{k=0}^{\infty} \frac{(-1)^{vk}}{\Gamma(vk+1)} (ct)^{vk} \tag{7}$$

Further Saxena, Mathai and Haubold [20] studied the generalizations of the fractional kinetic equation in terms of the Mittag-Leffler functions which extended the work of Haubold and Mathai [7]. In an another paper Saxena, Mathai and Haubold [21] developed the solutions for fractional kinetic equations associated with the generalized Mittag-Leffler function and R-function.
In the present paper we introduce and investigate the further computable extensions of the generalized fractional kinetic equation. The fractional kinetic equation and its solution, discussed in terms of the generalized Miller-Ross, are written in compact and easily computable form.

2. The New generalized Miller-Ross function:

This function introduced by the authors is defined as follows:

$$N_{p,q}^{\alpha,\beta,a}\left(a_1 \ldots . a_p;, b_1 \ldots . b_q; x\right) = \sum_{n=o}^{\infty} \frac{(a_1)_{nk} \ldots . (a_p)_{nk}}{(b_1)_{nk} \ldots . (b_q)_{nk}} \frac{a^n x^{n+\beta}}{\Gamma(\alpha n + \beta + 1)} \tag{8}$$

Here, p upper parameters $a_1, a_2, \ldots\ldots a_p$ and q lower parameters $b_1, b_2, \ldots . b_q$, $\alpha, \beta \epsilon C$, $R(\alpha) > 0$, $R(\beta) > 0$ and $(a_j)_{nk}$ $(b_j)_{nk}$ are pochammer symbols. The function (3) is defined when none of the denominator parameters $b_j s,\ j = 1,2, \ldots\ldots\ldots q$ is a negative integer or zero. If any parameter a_j is negative then the function (3) terminates into a polynomial in x. By using ratio test, it is evident that function (3) is convergent for all x, when $q \geq p$, it is convergent for $|x| < 1$ when $p = q + 1$, divergent when $p > q + 1$. In some cases the series is convergent for $x = 1,\ x = -1$. Let us consider take,

$$\beta = \sum_{j=1}^{p} a_j - \sum_{j=1}^{q} b_j$$

when $p = q + 1$, the series is absolutely convergent for $|x| = 1$ if $R(\beta) < 0$, convergent for $x = -1$, if $0 \leq R(\beta) < 1$ and divergent for $|x| = 1$, if $1 \leq R(\beta)$.

3. Generalized Fractional Kinetic Equations:

'In this section we investigate the solution of generalized fractional kinetic equation'. The results are obtained in a compact form in terms of New Generalized Miller-Ross function and are suitable for computation. The result is presented in the form of a theorem as follows:

Theorem 1:

If $v > 0, c > 0, \mu > 0$, then for the solution of the generalized fractional kinetic equation

$$N(t) - N_0 t^{-\mu(v-1)} N_{p,q}^{v,\mu,a} \left(a_1 \ldots . a_p; , b_1 \ldots . b_q; t^v\right) = -c^v {}_0D_t^{-v} N(t) \tag{9}$$

there holds the formula

$$N(t) = N_0 t^{-\mu(v-1)} \sum_{k=0}^{\infty} (-1)^k (ct)^{kv} \quad N_{p,q}^{2v,\mu,a} \left(a_1 \ldots . a_p; , b_1 \ldots . b_q; t^v\right) \tag{10}$$

Proof. Applying the Laplace transform both the sides of equation (9), we get

$$L\{N(t)\} - L\left\{N_0 t^{-\mu(v-1)} N_{p,q}^{v,\mu,a} \left(a_1 \ldots . a_p; , b_1 \ldots . b_q; t^v\right)\right\} = L\{-c^v {}_0D_t^{-v} N(t)\}$$

$$N(s) - N_0 \sum_{k=o}^{\infty} \frac{(a_1)_{nk} \ldots . (a_p)_{nk}\, a^k}{(b_1)_{nk} \ldots . (b_q)_{nk}} \frac{1}{s^{vk+\mu+1}} = -c^v s^{-v} N(s) \tag{11}$$

Solving for $N(s)$, it gives

$$N(s) = \frac{N_0}{(1 + c^v s^{-v})} \sum_{k=o}^{\infty} \frac{(a_1)_k \ldots . (a_p)_k\, a^k}{(b_1)_k \ldots . (b_q)_k} \frac{1}{s^{vk+\mu+1}} \tag{12}$$

Now, taking inverse Laplace transform both the sides of (12), we get

$$L^{-1}\{\overline{N}(s)\} = L^{-1}\left\{N_0 \sum_{k=o}^{\infty} \frac{(-1)^k (c^v s^{-v})^k (1)_k}{k!} \sum_{k=o}^{\infty} \frac{(a_1)_{nk} \ldots . (a_p)_{nk}\, a^k}{(b_1)_{nk} \ldots . (b_q)_{nk}} \frac{1}{s^{vk+\mu+1}}\right\} \tag{13}$$

Or

$$N(t) = N_0 t^{-\mu(v-1)} \sum_{k=0}^{\infty} (-1)^k (ct)^{kv} N_{p,q}^{2v,\mu,a} \left(a_1 \ldots . a_p; , b_1 \ldots . b_q; t^v\right) \tag{14}$$

This is complete proof of the statement (9).

4. Special Cases:

When $a = 1, \mu = 0$, then

Corollary: 1. If $v > 0, c > 0, \mu > 0$, then for the solution of the generalized fractional kinetic equation (in form of M-Series [22, 23])

$$N(t) - N_0 M_{p,q}^{v} \left(a_1 \ldots . a_p; , b_1 \ldots . b_q; t^v\right) = -c^v {}_0D_t^{-v} N(t) \tag{15}$$

there holds the formula

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k (ct)^{kv} M_{p,q}^{2v} \left(a_1 \ldots . a_p; , b_1 \ldots . b_q; t^v\right) \tag{16}$$

When $\mu = 0$, $a = 1$ and no upper and lower parameter then

Corollary: 2. If $v > 0, c > 0, \mu > 0$, then for the solution of the generalized fractional kinetic equation (in form of Mittag-Leffler function [10])

$$N(t) - N_0 E_v(t^v) = -c^v {}_0D_t^{-v} N(t) \tag{17}$$

there the relation

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k (ct)^{kv} \; E_v(t^v) \tag{18}$$

When $v = 1$ and no upper and lower parameter then

Corollary: 3. If $v > 0, c > 0, \mu > 0$, then for the solution of the generalized fractional kinetic equation (in form of Miller-Ross function [27])

$$N(t) - N_0 E_t(\mu, a) = -c^1 {}_0D_t^{-1} N(t) \tag{19}$$

there exist the result

$$N(t) = N_0 \sum_{k=0}^{\infty} (-1)^k (ct)^k E_t(\mu, a) \tag{20}$$

4. Conclusion:

In this paper we have introduced an extended fractional generalization of the standard kinetic equation and established solution for the same. Fractional kinetic equation can be used to compute the particle reaction rate and describes the statistical mechanics associated with the particle distribution function. The generalized fractional kinetic equation discussed in this article, involving New Generalized Miller-Ross function contains a number of known (may be new also) fractional kinetic equations involving various other special functions (the M-series, Mittag-Leffler function etc.).

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